# Analysis II Exam Cheatsheet v3

Ruben Schenk

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# 1 Ordinary Differential Equations

## 1.1 Introduction

**Definition (Differential equation):** A *differential equation* is an equation for a function $f$ that relates values of $f$ at $x$, $f(x)$, to the values of its derivatives at the same point $f'(x)$, $f''(x), ...$

**Definition (Order):** The *order* of a differential equation is the largest derivative present in the equation.

**Remark:** The solution of an *ordinary* differential equation is a function of only one variable $x$. A solution to a *partial* differential equation has several different variables.
**Remark:** In general the solution to an ODE is *not unique.* But if we are given "initial conditions", then we will be able to find unique solutions.
**Remark:** An ODE of order $k$ needs $k$ intial conditions to determine all $k$ constants of the unique solutions.

**Theorem:** Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of two variables. Let $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^2$. Then the ordinary differential equation

$$y' = F(x, y)$$

has a unique solution $f$ defined on a "largest" open interval $I$ containing $x_0$ such that $f(x_0) = y_0$. In other words, there exists $I$ and a function $f : I \to \mathbb{R}$ such that for all $x \in I$, we have $f'(x) = F(x, f(x))$, and one cannot find a larger interval containing $I$ with such a solution.

**Counter example:** Consider $2yy' = 1$ with $y(0) = 1$. Writing this as $(y^2)' = 1$, we see that $y^2 = x + a$ satisfies this eqn. With the initial condition, it follows that $y = f(x) = \sqrt{x + 1}$. Although the solution can be asked for all $x \in \mathbb{R}$, the solution only makes sense for $x > -1$.

## 1.2 Linear Differential Equations

**Definition (Linear ODE):** A *linear ODE* of order $k$ in interval $I \subset \mathbb{R}$ is an equation of the form

$$(*)\ y^{(k)} + a_{k-1}(x)y^{(k-1)} + ... + a_1(x)y' + a_0(x)y = b(x)$$

where $a_j(x)$ and $b(x)$ are continuous functions from $I$ to $\mathbb{C}$. If $b(x) = 0$, we say that the equation is *homogenous*, otherwise we say it is *inhomogeneous.*

**How to recognise a linear ODE:**

- no coefficients in front of highest derivative
- other coefficients are *continous* functions
- there are no products of $y$ and its derivatives
- neither $y$ nor its derivative occur to any power other than 1
- neither $y$ nor its derivative are "inside" another function, e.g. $\sqrt{y}$, $\sin(y)$

**Definition (Initial condition):** An *initial condition* for $(*)$ is a set of equations $y(x_0) = y_0$, $y'(x_0) = y_1, ...$ specifying values of $y$, $y', ...$ at some initial point $x_0$.

**Main results for linear ODE's:**
Let $I \subset \mathbb{R}$ be an open interval and $k \geq 1$ an integer and $\mathcal{S}_0 := \{f_h : I \to \mathbb{R} \,|\, f_h^{(k)}(x) + a_{k-1}f_h^{(k-1)}(x) + \dots + a_0 f_h(x) = 0\}$.

1. $\mathcal{S}_0$ is a vector space of dimension $k$, and $k$ is also the order of the ODE.
2. For any initial conditions $y_0, ..., y_{k-1} \in \mathbb{C}^k$ there is a unique solution $f_h \in \mathcal{S}_0$ such that $f_h(x_0) = y_0$, $f_h'(x_0) = y_1, ..., f_h^{(k-1)}(x_0) = y_{k-1}$.
3. For an arbitrary $b$, the set of solutions of the ODE has the form $\mathcal{S}_b = \{f_h + f_p \,|\, f_h \in \mathcal{S}_0\}$ where $f_p$ is one particular solution of the ODE.
4. For any inital condition there is a unique solution $f \in \mathcal{S}_b$.

**Remark:** If $b \neq 0$, the set $\mathcal{S}_b$ of solutions is *not* a vector space.
**Remark:** If we know a function $f_1$ solving a linear ODE with right-hand side $b_1$, and one function $f_2$ solving a linear ODE with right-hand side $b_2$, then $f_1 + f_2$ solves the linear ODE with right-hand side $b_1 + b_2$, since $D(f_1 + f_2) = D(f_1) + D(f_2) = b_1 + b_2$.

## 1.3 Linear Differential Equations of Order 1

We consider an ODE of the form $y' + ay = b$ where $a$, $b$ are continuous functions. We solve such an equation in two steps:

1. Find the solution $f_h$ of the corresponding hom. eqn. $y' + ay = 0$ such that $f_h' + af_h = 0$
2. Find a particular solution $f_p$ such that $f_p' + af_p = b$

### 1.3.1 Step 1: Solving the homogeneous equation

**Proposition:** Any solution of $y' + ay = 0$ is of the form $f_h(x) = z\exp(-A(x))$ where $A$ is the primitive of $a$. The unique solution with $f_h(x_0) = y_0$ is

$$f_h(x) = y_0 \exp(A(x_0) - A(x)).$$

**Recipe:** Assume $y' + ay = 0$:
$y' = \frac{dy}{dx} = -ay$ ($y = 0$ is a sol. so let $y \neq 0$) $\Leftrightarrow \int \frac{dy}{y} = -\int a dx$ (integrate) $\Leftrightarrow \ln|y(x)| = -A(x) \Leftrightarrow y(x) = z \cdot \exp(-A(x))$.

### 1.3.2 Step 2: Solving the inhomogeneous equation

There are three methods to solve an inhom. eqn. of the form $y' + ay = b$:
**Method 1: "Educated guess"**
If $b(x)$ is a polynomial, we guess that $f_p$ is also a polynomial. Same holds for $b(x)$ being a trigonometric function, an exponential function etc.
**Method 2: "Variation of constants"**
This approach uses the following two steps:

1. Assume $f_p = z(x)\exp(-A(x))$ for some function $z : I \to \mathbb{C}$.
2. We put this into the equation $y' + ay = b$ and see what this forces $z(x)$ to satisfy.

When putting $f_p = z(x)\exp(-A(x))$ into $y' + ay = b$, we see that it must hold that

$$y' + ay = z'(x)\exp(-A(x)) = b(x)$$

from where we arrive at the following equation: $z'(x) = b(x)\exp(A(x))$.
Hence $z(x) = \int dx\, b(x)\exp(A(x))$ and

$$f_p(x) = \Big(\int b(x)e^{A(x)}\Big)e^{-A(x)}.$$

**Method 3: "Integrating Factor Method"**
We proceed with the following three steps:

1. Determine the integrating factor (IF): $\exp(\int a(x)\, dx) = \exp(A(x))$
2. Multiply the ODE with the IF: $\exp(A(x))(\frac{dy}{dx} + a(x)y) = b(x)\exp(A(x)) \Rightarrow \frac{d}{dx}(y \cdot \exp(A(x))) = b(x)\exp(A(x))$ (integrate)
3. We arrive at the same solution as method 2: $z(x) = y \cdot \exp(A(x)) \Rightarrow y = z(x)\exp(-A(x))$ and $z'(x) = b(x)\exp(A(x)) \Rightarrow z(x) = \int dx\, b(x)\exp(A(x))$

**Example:** Let $(*) : y' + 2y = x$. We start with the homo. sol. $f_h$ : $y' + 2y = 0 \Rightarrow y' = -2y \Rightarrow \frac{dy}{dx} = -2y \Rightarrow \int \frac{dy}{y} = \int -2dx \Rightarrow \log(y) = -2x + c \Rightarrow y = ze^{-2x}$. We have for the homo. sol. $y_h = ze^{-2x}$. For the particular sol. we can choose either method.
Method 1: We assume $y_p$ to be of the form $b(x) = b$, i.e. $y_p = c_1x + c_2$. We put this into $(*)$ to get $2c_1x + c_1 + 2c_1 = x$. We see that $2c_1 = 1$ and $c_1 + 2c_2$, i.e. $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{4}$. It follows that $f_p = \frac{1}{2}x - \frac{1}{4}$.
Method 2: We state $y_p = z(x)e^{-2x}$ and put this into $(*)$ to get $z'(x) = xe^{2x}$. With $z(x) = \int xe^{2x}\, dx$ from where it follows that $z(x) = (\frac{1}{2}x - \frac{1}{4})e^{2x}$. Then $y_p = \frac{1}{2}x - \frac{1}{4}$.
Hence the general solution is $y = y_h + y_p = ze^{-2x} + \frac{1}{2}x - \frac{1}{4}$.

## 1.4 Linear Differential Equations with Constant Coefficients

We consider an ODE of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \dots + a_1y' + a_0y = b$ where $k \geq 1$ is an integer and $a_0, ..., a_{k-1} \in \mathbb{C}$ are constant coefficients. The right hand side $b$ is still assumed to be a continuous function.

### 1.4.1 Homogeneous solution

We assume to solution to be of the form $f_h(x) = e^{\lambda x}$. We then put this into the hom. eqn. to get $e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + \dots + a_0) = 0$.

**Definition (Characteristic Polynomial):** For a linear ODE with constant coefficients of the form $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$, the polynomial

$$P(\lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0$$

is called the *companion* or *characteristic polynomial* of the homogeneous differential equation. The zeroes of $P(\lambda)$ are called the *eigenvalues*.

**Remark:** The companion is only defined if the coefficients are constants!

**Theorem:** Let $\lambda_1, \ldots, \lambda_r$ be pairwise distinct eigenvalues of $P(\lambda)$, the characteristic polynomial of an ODE $(*)$ with corresponding multiplicities $m_1, \ldots, m_r$.
Then the functions

$$f_{j,l} : \mathbb{R} \to \mathbb{C},\ x \to x^l e^{\lambda_j x}$$

form $1 \leq j \leq r$ and $0 \leq l \leq n_j$ for a system of solutions of the hom. $(*)$

**Example:** $y'' - 2y' + 1 = 0$. Then $P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. Therefore $\lambda = 1$ has multiplicity of 2 and we have the two solutions $e^x$ and $xe^x$.
**Remark:** If $\lambda = \beta + i\gamma$ is a complex root of $P(\lambda)$, then so is $\bar{\lambda} = \beta - i\gamma$. Hence $f_1 := e^{\lambda x}$, $f_2 := e^{\bar{\lambda} x}$ are two solutions, where $e^{\lambda x} = e^{\beta x} \cdot e^{i\gamma x} = e^{\beta x}(\cos(\gamma x) + i\sin(\gamma x))$ and we can replace any solution with a lin. combination of $\tilde{f}_1 = e^{\beta x}\cos(x)$ and $\tilde{f}_2 = e^{\beta x}\sin(x)$.

**Theorem:** If $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ has real coefficients, then each pair of complex conjugate roots $\beta_j \pm i\gamma_j$ of $P(\lambda)$ with multiplicity $m_j$ leads to the solutions

$$x^l e^{\beta_j x}(\cos(\gamma_j x) + i\sin(\gamma_j x))$$

for $0 \leq l < m_j$, which then can be replaced by the solutions $x^l e^{\beta_j x}\cos(\gamma_j x)$ and $x^l e^{\beta_j x}\sin(\gamma_j x)$.

**Example:** $y^{(4)} + 2y'' + y = 0$. Then $P(\lambda) = \lambda^4 + 2\lambda^2 + 1 = 0 \Leftrightarrow (\lambda^2+1)^2 = 0 \Leftrightarrow (\lambda - i)^2 \cdot (\lambda + i)^2 = 0$. We therefore have the eigenvalues $i$, $-i$, both with multiplicity 2. Our solutions are $e^{ix}$, $xe^{ix}$, $e^{-ix}$ and $xe^{-ix}$ (which is equal to $\cos(x)$, $x\cos(x)$, $\sin(x)$ and $x\sin(x)$). We have the hom. solution $y_h := z_1 e^{ix} + z_2 xe^{ix} + z_3 e^{-ix} + z_4 xe^{-ix}$.
**Remark:** $Ae^{(\beta+i\gamma)} + Be^{\beta - i\gamma} :\Rightarrow \tilde{A}e^{\beta x}\cos(x) + \tilde{B}e^{\beta x}\sin(x)$, where $\tilde{A} = A + B$ and $\tilde{B} = i(A - B)$.

**Theorem:** $y = e^{\lambda x}$ is a solution of $(*)$ with $b = 0 \Leftrightarrow \lambda$ is an eigenvalue.

### 1.4.2 Inhomogeneous solution

We look at a inhom. eqn. of the form $(*)\, y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = b(x)$. The goal is to find a particular solution $y_p$, then any solution of $(*)$ is of the form $y = y_h + y_p$.
**1.4.2.1 Method 1: "Ansatz" or "Method of undetermined coefficients"**
The key idea is that the solution will be "similar" to the disturbance function $b(x)$.

| **b(x)** | **Ansatz** |
|---|---|
| $ae^{\alpha x}$ | $be^{\alpha x}$ |
| $a\sin(\beta x)$ or $b\cos(\beta x)$ | $c\sin(\beta x) + d\cos(\beta x)$ |
| $ae^{\alpha x}\sin(\beta x)$ or $be^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}(c\sin(\beta x) + d\cos(\beta x))$ |
| $P_n(x)e^{\alpha x}$ | $R_n(x)e^{\alpha x}$ |
| $P_n(x)e^{\alpha x}\sin(\beta x)$ or $Q_n(x)e^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}(R_n(x)\sin(\beta x) + S_n(x)\cos(\beta x))$ |

Solving $(*)$ with this method includes the following steps. Assume $b(x) = ae^{\alpha x}$:

1. We'll try $y_p = Be^{\alpha x}$ for some $B$.
2. We put $y_p$ into the diff. eqn. $(*)$. This will give us some conditions on $B$.
3. We solve for $B$.

**Remark:** If $b(x)$ is a lin. combination of some of the above functions, we try the corresponding lin. combination of the "Ansatz" functions.
**Remark:** If $\lambda = \alpha + i\beta$ is a zero of the char. pol. $P(\lambda)$ with multiplicity $m$, then the "Ansatz" function must be multiplied with $x^m$.
**Remark:** If the "Ansatz" is already a solution for $y_h$, we too multiply the "Ansatz" by $x$.
**Example:** $(*)\, y'' + y' - 6y = 10e^{2x}$. Then $e^{2x}$ is also a sol. to the homo. eqn. (and 2 is an eigenvalue of $P(\lambda)$ with mult. 1). Therefore we should try as Ansatz $kxe^{2x}$.
**Example:** $(*)\, y'' + y' - 6y = 3e^{-4x}$. Then $P(\lambda) = \lambda^2 + \lambda - 6 = 0 \Leftrightarrow (\lambda - 2) \cdot (\lambda + 3) = 0$. We have the roots $\lambda_1 = 2$, $\lambda_2 = -3$ and $y_h = z_1 e^{2x} + z_2 e^{-3x}$. To find $y_p$, we take as "Ansatz" $y_p = Be^{-4x}$ and put $y_p$ into $(*)$. We get that $16Be^{-4x} - 4Be^{-4x} - 6Be^{-4x} = 3e^{-4x} \Leftrightarrow 6Be^{-4x} = 3e^{-4x} \Rightarrow B = \frac{1}{2}$. It follows that $y_p = \frac{1}{2}e^{-4x}$. The general solution of $(*)$ is therefore given by $y = z_1 e^{2x} + z_2 e^{-3x} + \frac{1}{2}e^{-4x}$.
**1.4.2.2 Method 2: Variation of constants (2nd order)**
We use this method if $k = 2$ and the RHS $b(x)$ is "ugly".
Assume a diff. eqn. of the form $y'' + a_1 y' + a_0 y = b(x)$. We assume the hom. solution to be of the form $y_h = z_1 f_1 + z_2 f_2$, where $f_1$ and $f_2$ are two lin. indep. solutions. For $f_p$ we try a similar approach, namely $f_p = z_1(x)f_1 + z_2(x)f_2$, with the only difference that $z_1(x)$ and $z_2(x)$ are both functions of $x$ and not constants.
To determine $z_1(x)$ and $z_2(x)$ we need two equations, namely:

- $z_1'(x)f_1 + z_2'(x)f_2 = 0$
- $z_1'(x)f_1' + z_2'(x)f_2' = b(x)$

Once we found both $z_1'(x)$ and $z_2'(x)$ (see *Matrix formula below*), we can integrate both to find $y_p = z_1(x)f_1 + z_2(x)f_2$.
**Example:** Let $(*) : y'' - 2y' = \exp(x)$.
1) Find homo. soln.: $P(\lambda) = \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = 2$. It follows that $y_h = z_1 \cdot e^{0x} + z_2 e^{2x} = z_1 + z_2 e^{2x}$.
2) Find inhomo. soln.: We first state $f_1 = 1$, $f_2 = e^{2x} \Rightarrow f_1' = 0$, $f_2' = 2e^{2x}$. We need to satisfy the following two eqns.:

- $z_1'(x) \cdot 1 + z_2'(x) \cdot e^{2x} = 0$
- $z_1'(x) \cdot 0 + z_2'(x) \cdot 2e^{2x} = b(x) = e^x$

It follows that $z_2'(x) = \frac{e^{-x}}{2}$ and from this $z_1'(x) = -\frac{e^{-x}}{2}e^{2x} = \frac{-e^x}{2}$.
3) Integrate: $z_1 = \frac{-e^x}{2}$, $z_2 = \frac{-e^{-x}}{2}$.
4) Find $y_p$: $y_p = \frac{-e^x}{2} \cdot 1 + (\frac{-e^{-x}}{2}) \cdot e^{2x} = \frac{-e^x - e^x}{2} = -e^x$.
5) Verify the solution by putting it into $(*)$.
6) General solution is given by $y = y_h + y_p = c_1 \cdot 1 + c_2 \cdot e^{2x} - e^x$, $c_1, c_2 \in \mathbb{R}$.

**Matrix formula:** The system of eqns. from above can be rewritten as:

$$(A :=) \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \cdot \begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ b(x) \end{pmatrix}.$$

As $A$ is invertible we have

$$\begin{pmatrix} z_1'(x) \\ z_2'(x) \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} 0 \\ b(x) \end{pmatrix}.$$

### 1.4.3 Separation of variables

Used if $k = 1$ and there is a product of 2 functions where one relies on $x$ and the other on $y$. The ODE doesn't need to be linear.

**Definition (Separable ODE):** A diff. eqn. of first order is called *separable* if it is of the form $y' = b(x)g(y)$.
We can separate the variables $x$ and $y$:

$$\frac{dy}{dx} = b(x)g(y) \Leftrightarrow \int \frac{dy}{g(y)} = \int b(x)dx,$$

which results in the LHS being some function of $y$ and the RHS being some function of $x$.

**Remark:** For any $y_0$ such that $g(y_0) = 0$, the constant function $y = y_0$ is also a solution.
**Example:** $e^{2y}y' = x \Rightarrow y' = xe^{-2y}$. In this case, $b(x) = x$ and $g(y) = e^{-2y}$. Note that $g(y)$ is never zero. We then have $\int e^{2y}\, dy = \int x\, dx \Rightarrow \frac{e^{2y}}{2} = \frac{x^2}{2} + C$. From here it follows that: $e^{2y} = x^2 + C' \Leftrightarrow 2y = \log(x^2 + C) \Leftrightarrow y = \frac{\log(x^2+C)}{2}$.

# 2 Differential Calculus in $\mathbb{R}^n$

## 2.1 Introduction

**Remark:** Any function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a *Cartesian product* of functions $f_1, ..., f_m$ where each $f_i : \mathbb{R}^n \to \mathbb{R}$ is just the $i$-th component function of $f$.

## 2.2 Continuity in $\mathbb{R}^n$

**Recall:** We say that $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $x_0 \in \mathbb{R}$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

**Definition (Convergence):** Let $(x_k)_{k\in\mathbb{N}}$ where $x_k \in \mathbb{R}^n$. We write $x_k = (x_{k,1}, ..., x_{k,n})$.
Let $y = (y_1, ..., y_n) \in \mathbb{R}^n$. We say that the sequence $(x_k)$ *converges* to $y$ as $k \to \infty$ if for all $\epsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$, we have $||x_k - y|| < \epsilon$.

**Lemma:** The above definition is equivalent to each of the following conditions:

1. For each $1 \leq i \leq n$, the sequence $(x_{k,i})$ of real numbers converges to $y_i$.
2. The sequence of real numbers $||x_k - y||$ converges to 0 as $k \to \infty$.

**Definition (Continuity):** Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$.
(1) Let $x_0 \in X$. We say that $f$ is *continuous* at $x_0$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x \in X$ satisfies $||x - x_0|| < \delta$, then

$$||f(x) - f(x_0)|| < \epsilon.$$

(2) We say that $f$ is *continuous on* $X$ if it is continuous at $x_0$ for all $x_0 \in X$.
(3) $f$ is continuous at $x_0 \Leftrightarrow \lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x) = f(x_0)$.

**Proposition (Test continuity using sequences):** The function $f$ is *continuous* at $x_0$ if and only if, for every sequence $(x_k)_{k\geq 1} \in X$ such that $x_k \to x_0$ as $k \to \infty$, the sequence $(f(x_k)_{k\geq 1}) \in \mathbb{R}^m$ converges to $f(x_0)$, i.e. $\lim f(x_k) = f(\lim x_k)(= f(x_0))$.

**Definition (Limit):** Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let $x_0 \in X$ and $y \in \mathbb{R}^m$. We say that $f$ *has the limit* $y$ as $x \to x_0$ with $x \neq x_0$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in X$, $x \neq x_0$, such that $||x - x_0|| < \delta$, we have $||f(x) - f(x_0)|| < \epsilon$. We then write

$$\lim_{x\to x_0,\, x\neq x_0} f(x) = y.$$

**Proposition:** We have $\lim_{x\to x_0} f(x) = y$ if and only if for every sequence $(x_k) \in X$ such that $x_k \to x$ as $k \to \infty$, and $x_k \neq x_0$, the sequence $(f(x_k)) \in \mathbb{R}^m$ converges to $y$.
**Remark:** The above proposition means that we can use the convergence and the limit of a function interchangeably. **Examples:**

1. $f : \mathbb{R}^n \to \mathbb{R}^m$, $x = (x_1, ..., x_n) \to (f_1(x), ..., f_m(x))$ and $f_i : \mathbb{R}^n \to \mathbb{R}$. Then $f$ is continuous at $x_0 \Leftrightarrow f_i$ is continuous at $x_0$ $\forall i = 1, ..., m$.
2. Linear functions, i.e. $f : \mathbb{R}^n \to \mathbb{R}^m$, $x \to Ax$, are continuous.
3. Polynomials are continuous.
4. Sums and products of continuous functions are continuous.
5. Functions of separated variables are continuous if the factors are continuous.
6. Compositions of continuous functions are continuous but the opposite must not be true.
7. $[a, b) \subset \mathbb{R}$ is neither closed nor open.

**Example:** We define $f(x\,y) = x$, if $y > 0$, $-x$, if $y \leq 0$. For any fixed $y = y_0$, $g_{y_0}(x) = g(x) = f(x, y_0)$ is continuous but $f$ as a function of two variables is not.

**Sandwich lemma for $f : \mathbb{R}^n \to \mathbb{R}$**
Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) < g(x) < h(x) \forall x \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

**Definition:**
(1) A subset $X \subset \mathbb{R}^n$ is *bounded* if the set of $||x||$ for $x \in X$ is bounded in $\mathbb{R}$.
(2) A subset $X \subset \mathbb{R}^n$ is *closed* if for every sequence $(x_k)$ in $X$ that converges in $\mathbb{R}^n$ to some vector $y \in \mathbb{R}^n$, we have $y \in X$.
(3) A subset $X \in \mathbb{R}^n$ is *compact* if it is bounded and closed.

**Examples:**

1. $\emptyset$ and $\mathbb{R}^n$ are both closed.
2. $B_r(x_0) := \{x \in \mathbb{R}^n \,|\, ||x - x_0|| < r\}$ is bounded but not closed, because $(x_k) = x_0 + (r - \frac{1}{k}, 0, ..., 0)^T \subset B_r(x_0)$ (therefore bounded) but $\lim_{k\to\infty}(x_k) = x_0 + (r, 0, ..., 0)^T \notin B_r(x_0)$ (therefore not closed).
3. $\left(\frac{1}{k}\right)_k \subset (0, 1] = X \subset \mathbb{R}$. Now $\frac{1}{k} \to 0$ but $0 \notin X$ and therefore not closed.
4. $A = I_1 \times \cdots \times I_n = \{(x_1, ..., x_n) \in \mathbb{R}^n \,|\, x_i \in I_i\}$, $I_i = [a_i, b_i]$. Then $A$ is bounded and closed, hence compact in $\mathbb{R}^n$.

**Remark:** If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are bounded (resp. closed / resp. compact), then $X \times Y := \{(x, y) \in \mathbb{R}^{n+m} \,|\, x \in X, y \in Y\}$ is bounded (resp. closed / resp. compact).
**Proposition:** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. For any closed set $Y \subset \mathbb{R}^m$, the set $f^{-1}(Y) = \{x \in \mathbb{R}^n \,|\, f(x) \in Y\} \subset \mathbb{R}^n$ is closed.

**Definition: Partial integration**
$\int f'(x) \cdot g(x)dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x)dx$

**Remark:** The inverse image of closed sets under continuous maps are closed, example:

- If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then for any $a \leq b$, $X = \{x \in \mathbb{R}^n \,|\, a \leq f(x) \leq b\}$ is closed and $X = f^{-1}([a, b])$.

But we cannot conclude that it's also compact, since it might be not bounded, eg.

- $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \to \sin(xyz)$, then $\{(x, y, z) \,|\, -1 \leq f(x) \leq 1\} = f^{-1}([-1, 1]) = \mathbb{R}^3$ is closed but not bounded.

**Min-Max Theorem for Functions of Several Variables**
Let $X \subset \mathbb{R}^n$ be a non-empty compact set and $f : X \to \mathbb{R}$ a continuous function. Then $f$ is bounded and achieves its maximum and minimum, or in other words, there exists $x_+$ and $x_-$ in $X$ such that

$$f(x_+) = \sup_{x\in X} f(x) \text{ and } f(x_-) = \inf_{x\in X} f(x).$$

## 2.3 Partial derivatives

**Definition (Open subset):** A subset $X \subset \mathbb{R}^n$ is *open* if, for any $x = (x_1, ..., x_n) \in X$, there exists $\delta > 0$ such that the set

$$\{y = (y_1, ..., y_n) \in \mathbb{R}^n \,|\, |x_i - y_i| < \delta \,\forall i\}$$

is contained in $X$. In other words. any point of $\mathbb{R}^n$ obtained by changing any coordinate of $x$ by at most $\delta$ is still in $X$.
Furthermore, $X \subset \mathbb{R}^n$ is called *open* if its complement $\mathbb{R}^n \setminus X$ is closed.

**Corollary:** If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $Y \subset \mathbb{R}^m$ is open, then $f^{-1}(Y)$ is open in $\mathbb{R}^n$.

**Definition (Partial derivative):** Let $X \subset \mathbb{R}^n$ be an open set. Let $f : X \to \mathbb{R}^m$ be a function and let $1 \leq i \leq n$. We say that $f$ has a *partial derivative* on $X$ with respect to the $i$-th variable, or coordinate, if for all $x_0 = (x_{0,1}, x_{0,2}, ..., x_{0,n}) \in X$, the function defined by

$$g_i(t) = (x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n})$$

on the set

$$I = \{t \in \mathbb{R} \,|\, (x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n}) \in X\}$$

is differentiable at $t = x_{0,i}$. Its derivative $g_i'(x_{0,i})$ at $x_{0,i}$ is denoted

$$\frac{\partial f}{\partial x_i}(x_0),\; \partial_{x_i} f(x_0),\; \partial_i f(x_0).$$

**Remark:** $g_i'(x_{0,i}) = \frac{\mathrm{d}g_i}{\mathrm{d}t}(x_{0,i}) = \lim_{h\to 0} \frac{g_i(x_{0,i}+h) - g_i(x_{0,i})}{h}$.
**Remark:** To evaluate partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ with respect to $x_j$ at a point $a = (a_1, ..., a_n)$, we differentiate $f(a_1, ..., a_{j-1}, x_j, a_{j+1}, ..., a_n)$ with respect to $x_j$, treating all other variables as a constant with respect to $x_j$.

**Example:** $f:\mathbb{R}^2\to\mathbb{R}^3$, $(x,y)\to\begin{pmatrix}x^2+y^2\\2x\\2y\end{pmatrix}$, then $f_1:(x,y)\to x^2+y^2$, $f_2:(x,y)\to 2x$ and $f_3:(x,y)\to 2y$. The partial derivative with respect to $x$ is given by: $\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial f_1}{\partial x}\\\frac{\partial f_2}{\partial x}\\\frac{\partial f_3}{\partial x}\end{pmatrix}=\begin{pmatrix}2x\\2\\0\end{pmatrix}$.

**Definition (Jacobi matrix):** Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$ a function with partial derivatives on $X$. Write $f(x)=(f_1(x),...,f_m(x))$. For any $x\in X$, the matrix

$$J_f(x)=(\partial_{x_j}f_i(x))_{\substack{1\le i\le m\\1\le j\le n}}$$

with $m$ rows and $n$ columns is called the *Jacobi matrix* of $f$ at $x$.

**Definition (Gradient):** Let $X\subset\mathbb{R}^n$ be open and let $f:X\to\mathbb{R}$ be a function. If all partial derivatives of $f$ exist at $x_0\in X$, then the column vector

$$\begin{pmatrix}\partial_{x_1}f(x_0)\\\vdots\\\partial_{x_n}f(x_0)\end{pmatrix}$$

is called the *gradient* of $f$ at $x_0$ and is denoted $\nabla f(x_0)$.

**Proposition:** Consider $X\subset\mathbb{R}^n$ open and $f$, $g$ functions from $X$ to $\mathbb{R}^m$. Let $1\le i\le n$.
(1) If $f$ and $g$ have partial derivatives with respect to the $i$-th coordinate on $X$, then $f+g$ also does and $\partial_{x_i}(f+g)=\partial_{x_i}(f)+\partial_{x_i}(g)$.
(2) If $m=1$, and if $f$ and $g$ have partial derivatives with respect to the $i$-th coordinate on $X$, then $fg$ also does and $\partial_{x_i}(fg)=\partial_{x_i}(f)g+f\partial_{x_i}(g)$.
(3) Furthermore, if $g(x)\neq 0$ for all $x\in X$, then $\frac{f}{g}$ has a partial derivative with respect to the $i$-th coordinate on $X$, with $\partial_{x_i}(\frac{f}{g})=\frac{\partial_{x_i}(f)g-f\partial_{x_i}(g)}{g^2}$.

## 2.4 The differential

**Definition (Differentiable):** Let $X\subset\mathbb{R}^n$ open, $x_0\in X$ and $f:X\to\mathbb{R}^m$ a function. We say $f$ is *differentiable* at $x_0$ if there exists a linear map $u:\mathbb{R}^n\to\mathbb{R}^m$ such that

$$\lim_{\substack{x\to x_0\\x\neq x_0}}\frac{f(x)-f(x_0)-u(x-x_0)}{||x-x_0||}=0$$

where the limit is in $\mathbb{R}^m$. We denote $df(x_0)=u$. If $f$ is differentiable at every $x_0\in X$, we say $f$ is differentiable on $X$.

**Remark:** For $f:\mathbb{R}^n\to\mathbb{R}^m$ the total differential is not a number but a linear map!

**Theorem:** Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$ be a function that is differentiable on $X$.
1. The function $f$ is *continuous* on $X$.
2. The function $f$ admits partial derivatives on $X$ with respect to each variable.
3. Assume $m=1$. Let $x_0\in X$, and let $u(x_1,...,x_n)=a_1x_1+\cdots a_nx_n$ be the differential of $f$ at $x_0$. We then have
$$\partial_{x_i}f(x_0)=a_i$$
for $1\le i\le n$.

**Example:** $f:\mathbb{R}^3\to\mathbb{R}$, $(x,\,y,\,z)\to e^xy+zx$ and $x_0=(0,\,1,\,2)$. Then $\frac{\partial f}{\partial x}=e^xy+z$, $\frac{\partial f}{\partial y}=e^x$ and $\frac{\partial f}{\partial z}=x$. Now with $x_0=(0,\,1,\,2)$ we have that $\frac{\partial f}{\partial x}(x_0)=3$, $\frac{\partial f}{\partial y}(x_0)=1$ and $\frac{\partial f}{\partial z}=0$ and $J_f(x_0)=\begin{pmatrix}3&0&1\end{pmatrix}$. It follows: $df(x_0):\mathbb{R}^3\to\mathbb{R}$, $\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}3&1&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}$ and therefore

$$df(x_0)\begin{pmatrix}x\\y\\z\end{pmatrix}=3x+y.$$

**Proposition:** Let $X\subset\mathbb{R}^n$ be open, $f,\,g:X\to\mathbb{R}^m$ differentiable functions on $X$.
(1) The function $f+g$ is differentiable with differential $d(f+g)=df+dg$.
(2) If $m=1$, then $fg$ is differentiable.
(3) If $m=1$ and if $g(x)\neq 0$ for all $x\in X$, then $\frac{f}{g}$ is differentiable.

**Proposition:** Let $X\subset\mathbb{R}^n$ be open, $Y\subset\mathbb{R}^m$ be open and let $f:X\to Y$ and $g:Y\to\mathbb{R}^p$ be differentiable functions. Then $g\circ f:X\to\mathbb{R}^p$ is differentiable on $X$ and for any $x_0\in X$, its differential is given by the composition

$$d(g\circ f)(x_0)=dg(f(x_0))\circ df(x_0).$$

In particular, the Jacobi matrix satisfies

$$J_{g\circ f}(x_0)=J_g(f(x_0))J_f(x_0).$$

**Theorem:** If $f:X\to\mathbb{R}^m$, $X\subset\mathbb{R}^n$, has all partial derivatives $\frac{\partial f_i}{\partial x_j}:X\to\mathbb{R}^m$ and if these functions are *continuous* on $X$, then $f$ is *differentiable* on $X$.

**Definition (Tangent space):** Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$ a function that is differentiable. Let $x_0\in X$ and $u=df(x_0)$ be the differential of $f$ at $x_0$. The graph of the affine linear approximation

$$g(x)=f(x_0)+u(x-x_0)$$

from $\mathbb{R}^n$ to $\mathbb{R}^m$, or in other words the set $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m\,|\,y=f(x_0)+u(x-x_0)\}$, is called the *tangent space* at $x_0$ to the graph of $f$.

**Definition (Directional derivative):** Let $X\subset\mathbb{R}^n$ be an open set and let $f:X\to\mathbb{R}^m$ be a function. Let $v\in\mathbb{R}^m$ be a non-zero vector and $x_0\in X$. We say that $f$ has *directional derivative* $w\in\mathbb{R}^m$ in the direction $v$, if the function $g$ defined on the set

$$I=\{t\in\mathbb{R}\,|\,x_0+tv\in X\}$$

by

$$g(t)=f(x_0+tv)$$

has a derivative at $t=0$, and this is equal to $w$.
If $f$ is differentiable at $x_0$, then the directional derivative of $f$ at $x_0$ in the direction of $v$ exists.
We also say that $v\neq 0\in\mathbb{R}^n$ is called the *directional derivative* of $f$ along $v$ at a point $x_0\in\mathbb{R}^n$ if the following limit exists (and the derivative is equal to the limit):

$$D_vf(x_0)=\lim_{h\to 0}\frac{f(x_0+hv)-f(x_0)}{h}.$$

**Example:** Let $f(x,y)=(x^2+y^2,\,2x,\,2y)^T$. Then $J_f(1,2)=\begin{pmatrix}2&4\\2&0\\0&2\end{pmatrix}$ and $(df)(1,2):\mathbb{R}^2\to\mathbb{R}^3$, $(x,y)^T\to\begin{pmatrix}2&4\\2&0\\0&2\end{pmatrix}\cdot(x,\,y)^T$.
Therefore, the directional derivative of $f$ at $(1,2)$ in the direction of $\vec{v}=(1,\,1)^T$ is given by $(df(1,2))(1,\,1)^T=(6,\,2,\,2)^T$.
**Polar coordinates:** Let $f:[0,\,\infty)\times[0,\,2)\pi\to\mathbb{R}^2$, $(r,\phi)\to(x,y)=(r\cos(\phi),r\sin(\phi))$, i.e.

$$f(r,\phi)=\begin{pmatrix}f_1(r,\phi)\\f_2(r,\phi)\end{pmatrix}=\begin{pmatrix}r\cos(\phi\\r\sin(\phi))\end{pmatrix}.$$

The Jacobi matrix is given by

$$J_f(r,\phi)=\begin{pmatrix}\frac{\partial f_1}{\partial r}&\frac{\partial f_1}{\partial\phi}\\\frac{\partial f_2}{\partial r}&\frac{\partial f_2}{\partial\phi}\end{pmatrix}=\begin{pmatrix}\cos(\phi)&-r\sin(\phi)\\\sin(\phi)&r\cos(\phi)\end{pmatrix}.$$

## 2.5 Higher derivatives

**Definition (Smoothness):** Let $X\subset\mathbb{R}^n$ be open and $f:X\to\mathbb{R}^m$. We say that $f$ is of class $C^1$ if $f$ is differentiable on $X$ and all its partial derivatives are continuous. The set of functions of class $C^1$ from $X$ to $\mathbb{R}^m$ is denoted $C^1(X;\mathbb{R}^m)$.
Let $k\ge 2$. We say, by induction, that $f$ is of class $C^k$ if it is differentiable and each partial derivative $\partial_{x_i}f:X\to\mathbb{R}^m$ is of class $C^{k-1}$.
If $f\in C^k(X;\mathbb{R}^m)$ for all $k\ge 1$, then we say that $f$ is of class $C^\infty$.
If $f$ is in $C^\infty(X;\mathbb{R}^m)$, we say $f$ is *smooth*.

**Remark:** All polynomials, trigonometric functions and exponentials are smooth.

**Theorem (Mixed derivatives commute):** Let $k \geq 2$. Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}^m$ be a function of class $C^k$. Then the partial derivatives of order $k$ are independent of the order in which the partial derivatives are taken: for any variable $x$ and $y$ we have

$$\partial_{x,y} f = \partial_{y,x} f,$$

and for more variables the same holds.

**Definition (Hessian matrix):** Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a $C^2$ function. For $x_0 \in X$, the *Hessian matrix* of $f$ at $x_0$ is the symmetric $n \times n$ matrix

$$\text{Hess}_f(x_0) := \Big(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\Big)_{1 \leq i,j \leq n} = (\partial_{x_i,x_j} f)_{1 \leq i,j \leq n}.$$

We also write $H_f(x)$.

### 2.6 Taylor polynomials

**Definition (Taylor polynomial):** Let $k \geq 1$ be an integer. Let $f : X \to \mathbb{R}$ be a function of class $C^k$ on $X$, and fix $x_0 \in X$. The $k$-th *Taylor polynomial* of $f$ at the point $x_0$ is the polynomial in $n$ variables of degree $\leq k$ given by

$$\begin{aligned} T_k f(y; x_0) &= f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \cdots \\ &= \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0) y^m \end{aligned}$$

where the last sum ranges over the tuples of $n$ non-negative integers such that the sum is $k$.

**Theorem:** Let $k \geq 1$ be an integer. Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ be a function of class $C^k$. For $x_0 \in X$, if we define $E_k f(x; x_0)$ b¡ $f(x) = T_k f(x - x_0; x_0) + E_k f(x : x_0)$ then we have $\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{E_k f(x:x_0)}{||x - x_0||^k} = 0$.

**First order Taylor polynomial:**

$$T_1 f(y; x_0) = f(x_0) + \sum_{i=1}^n \partial_{x_i} f(x_0) y_i.$$

**Second order Taylor polynomial:**

$$\begin{aligned} T_2 f(y : x_0) = f(x_0) + \sum_{i=1}^n \partial_{x_i} f(x_0) y_i + \frac{1}{2} \sum_{i=1}^n \partial_{x_i^2}^2 f(x_0) y_i^2 \\ + \sum_{1 \leq i < j \leq n} \partial_{x_i x_j}^2 f(x_0) y_i y_j. \end{aligned}$$

### 2.7 Critical points and extremas

**Definition (Critical point):** Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a differentiable function. A point $x_0 \in X$ such that $\nabla f(x_0) = 0$ is called a *critical point* of the function $f$ If a critical points is neither a local maximum nor a local minimum, then it's called a *saddle point.*

**Proposition:** If $f : X \to \mathbb{R}$ differentiable on the interior of $X$ and $X$ is closed and bounded, then the *global extrema* of $f$ exists and it is either at a critical point of $f$ or on the boundary of $X$.

**Definition (Non-degenerate):** Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ a function of class $C^2$. A critical point $x_0 \in X$ of $f$ is called *non-degenerate* if the Hessian matrix has a non-zero determinant.

**Theorem:** Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ differentiable. We say $x_0 \in X$ is a *local maximum (local minimum)* if we can find a neighbourhood $B_r(x_0) = \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$, $B_r(x_0) \subset X$ such that

$$\forall x \in B_r(x_0)\ f(x) \leq f(x_0)\ (f(x) \geq f(x_0)).$$

If $x_0 \in X$ is a local extrema, then $\nabla f(x_0) = 0$, i.e. $\frac{\partial f}{\partial x_1}(x_0) = \cdots = \frac{\partial f}{\partial x_n}(x_0) = 0$.

**Recall:** A symmetric matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $\det(A) \neq 0$ is

1. positive definite if it's eigenvalues are only positive,
2. negative definite if it's eigenvalues are only negative,
3. indefinite if it's eigenvalues are both positive and negative.

**Proposition:** Let $f : X \to \mathbb{R}$, $f \in C^2(> X, \mathbb{R})$ and $X \subset \mathbb{R}^n$. Let $x_0 \in X$ be a non-degenerate critical point of $f$, i.e. $\nabla f(x_0) = 0$ and $\det Hess_f(x_0) \neq 0$. Then:

1. If $Hess_f(x_0) > 0$, then $x_0$ is a loc. min.
2. If $Hess_f(x_0) < 0$, then $x_0$ is a loc. max.
3. If $Hess_f(x_0)$ is indefinite, then $x_0$ is a saddle point.

If $\det(\text{Hess}_f(x_0) = 0)$, i.e. the point is degenerate, we cannot conclude anything about the point $x_0$: It could be a loc. min., loc. max. or even a saddle point!

### 2.8 The inverse and implicit functions theorems

**Definition:** Let $X \subset \mathbb{R}^n$ be an open set and $f : X \to \mathbb{R}^n$ differentiable. We say $f$ is a *change of variables* around $x_0$ if there is a radius $\rho > 0$ such that the restriction of $f$ to the ball around $x_0$ of radius $\rho$, $B_\rho := \{x \in \mathbb{R}^n \mid ||x - x_0|| < \rho\}$, so that the image $Y = f(B_\rho)$ is open in $\mathbb{R}^n$ and there exists a differentiable map $g : Y \to B$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_{B_\rho}$.

**Inverse Function Theorem:** Let $X \subseteq \mathbb{R}^n$ be open, $f : X \to \mathbb{R}^n$ differentiable. If $x_0 \in X$ is such that $\det(J_f(x_0)) \neq 0$, i.e. $J_f(x_0)$ is invertible, then $f$ is a change of variables around $x_0$. Moreover the Jacobian of $g$ at $x_0$ is defined by

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

## 3 Integration in $\mathbb{R}^n$

### 3.1 Line integrals

**Definition:** Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $f(t) = (f_1(t), ..., f_n(t))$ be a continuous function from $I$ to $\mathbb{R}^n$, i.e., $f_i$ is continuous for $1 \leq i \leq n$. Then we define

$$\int_a^b f(d)\, dt = \Big(\int_a^b f_1(t)\, dt, ..., \int_a^b f_n(t)\, dt\Big) \in \mathbb{R}^n.$$

**Definition (Line integral):** A *parameterized curve* in $\mathbb{R}^n$ is the image of a function $\gamma : [a, b] \to \mathbb{R}^n$ where the function is continuous and piecewise in $C^1$, i.e. there exists a partition $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ such that the restriction of $f$ to $]t_{j-1}, t_j[$ is $C^1$. With $f : X \to \mathbb{R}^n$, $X \subset \mathbb{R}^n$ a subset containing the image of $\gamma$, the integral

$$\int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt \in \mathbb{R}$$

is called the *line integral* of $f$ along $\gamma$. It is denoted

$$\int_\gamma f(s) \cdot ds, \text{ or } \int_\gamma f(s) \cdot d\vec{s}.$$

**Example:** Let $f(x, y) = (-y, x)$ and $\gamma(t) = (\cos(t), \sin(t))$, $t \in [0, 2\pi]$. Then $\int_\gamma f(s) \cdot ds = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\, dt$. Now with $\gamma'(t) = (-\sin(t), \cos(t))$, we get that $\int_0^{2\pi}(-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt = \int_0^{2\pi}(\sin^2(t) + \cos^2(t))\, dt = 2\pi$.

**Definition (Potential):** A differentiable scalar field $g : X \subset \mathbb{R}^n \to \mathbb{R}$ s.t. $\nabla g = f$, $f : X \to \mathbb{R}^n$, is called a *potential* for $f$.

**Remark:**

1. If $n = 1$, a potential of $f$ is the same as a primitive of $f$, i.e. $g$ such that $g' = f$.
2. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then $f$ always has a primitive, namely $g(x) := \int_a^x f(t)\, dt$.

**Definition (Conservative):** Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous vector field. If, for any $x_1, x_2 \in X$, the line integral $\int_\gamma f(s) \cdot ds$ is independent of the choice of a parameterized curve $\gamma \in X$ from $x_1$ to $x_2$, then we say that the vector field is *conservative.*
Equivalently, $f$ is conservative if and only if

$$\int_\gamma f(s) \cdot d\vec{s} = 0$$

for any *closed* parametrized curve in $X$ (i.e. $\gamma(a) = \gamma(b)$).

**Definition:** Let $X \subset \mathbb{R}^n$ be open. Then $X$ is said to be *path connected,* if for every pair of points $x, y \in X$, there exists a $C^1$-path $\gamma : (0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

**Theorem:** Let $f$ be a continuous vector field on an open path connected set $X$. Then the following are equivalent:

1. $f$ is the gradient of a function $g : X \to \mathbb{R}$, i.e. $f = \nabla g$, i.e. $g$ is a potential for $f$.
2. The line integral of $f$ is independent of the path between 2 points.
3. The line integral of $f$ around any closed curve is 0.

**Definition:** A subset $X \subset \mathbb{R}^n$ is called *star shaped* if $\exists x_0 \in X$ such that $\forall x \in X$, the line segment joining $x_0$ to $x$ is in $X$.

**Theorem:** Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ a $C^1$ vector field. If f is conservative, then

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$$

Warning: This is a one-directional implication, the other way around this doesn't hold.

**Theorem:** Let $X$ be a star shaped open subset of $\mathbb{R}^n$ and $f \in C^1$ a vector field such that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ $\forall i, j$, then $f$ is conservative.

**Definition (Curl):** Let $X \subset \mathbb{R}^3$ be open and $f \in C^1 : X \to \mathbb{R}^3$ a vector field. Then the *curl* of $f$ is the vector field on $X$ defined by

$$\text{curl}(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_3 \end{pmatrix}.$$

### 3.1.1 Properties of the line integral

1. It is independent of orientation preserving reparametrization of the curve.
I.e. if $\gamma[a,\, b] \to \mathbb{R}^n$ is a $C^1$ curve and let $\tilde{\gamma} : [c,\, d] \to \mathbb{R}^n$ such that $\tilde{\gamma} = \gamma \circ \phi$ where $\phi : [c,\, d] \to [a,\, b]$ in $C^1$ such that $\phi(c) = a$ and $\phi(d) = b$ with $\phi' > 0$ $\forall t \in [c,\, d]$ (and therefore orientation preserving), then

$$\int_\gamma f(s) \cdot ds = \int_{\tilde{\gamma}} f(s) \cdot ds.$$

2. Let $\gamma_1 : [a,\, b] \to \mathbb{R}^n$, $\gamma_2 : [c,\, d] \to \mathbb{R}^n$ 2 paths with $\gamma_1(b) = \gamma_2(c)$. We define $\gamma_1 + \gamma_2$ as the path formed by concatenation of the 2 curves. Then

$$\int_{\gamma_1+\gamma_2} f(s) \cdot ds = \int_{\gamma_1} f(s) \cdot ds + \int_{\gamma_2} f(s) \cdot ds.$$

3. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a path and $-\gamma : [a, b] \to \mathbb{R}^n$ be the same path traced in the opposite direction, i.e. $(-\gamma)(t) := \gamma(a + b - t)$. Then

$$\int_{-\gamma} f(s) \cdot ds = -\int_\gamma f(s) \cdot ds.$$

4. If $\exists g : X \to \mathbb{R}$, $g \in C^1$, such that $\nabla g = f$, then for any $\gamma : [a,\, b] \to \mathbb{R}$ with $\gamma([b,\, a]) \subset X$, we have

$$\int_\gamma f(s) \cdot ds = (g \cdot \gamma)(b) - (g \cdot \gamma)(a)$$

i.e. $\int_\gamma f(s) \cdot ds$ only depends on the endpoints of the curve.

## 3.2 The Riemann integral in $\mathbb{R}^n$

The analog to a closed interval in $\mathbb{R}^n$ is a *closed rectangle* $Q = I_1 \times \cdots \times I_n$ where $I_k = [a_k,\, b_k]$ and $Q = \{(x_1, ..., x_n) \in \mathbb{R}^n \,|\, x_k \in I_k\}$. We define the *volume* or *measure* of such a cube as $vol(Q) = \mu(Q) = \prod_{i=1}^n (b_i - a_i)$. A *partition* $P$ of $Q$ is a subcollection of rectangular boxes $Q_1, ..., Q_k \subset Q$ such that

1. $Q = \bigcup_{j=1}^k Q_j$
2. $\text{Interior}(Q_i) \cap \text{Interior}(Q_j) = \emptyset,\ i \neq j$

We define $\text{Norm}(P) = \delta_P := \max(\text{diameter}(Q_j))$. For each $Q_j$ we choose an intermediate point $\xi_j \in Q_j$. Then we define the *Riemann sum* of $f$, for partition $P$ and intermediate points $\{\xi_j\}$ as the sum

$$R(f,\, P,\, \xi) := \sum_{j=1}^k f(\xi_j) vol(Q_j).$$

We furthermore define the

- Upper Riemann sum $U_f(P) := \sum_{j=1}^k (\inf_{x \in Q_j} f(x)) vol(Q_j)$
- Lower Riemann sum $L_f(P) := \sum_{j=1}^k (\sup_{x \in Q_j} f(x)) vol(Q_j)$

Finally we can define the *Rieman integral* as

- Lower Riemann integral $\underline{I}(f) := \underline{\int_Q} f(x)\, dx := \sup\{L_f(P) \,|\, P \text{ runs over all partitions of } Q\}$
- Upper Riemann integral $\overline{I}(f) := \overline{\int_Q} f(x)\, dx := \inf\{U_f(P) \,|\, P \text{ runs over all partitions of } Q\}$

Then $f$ is called *integrable* if $\underline{I}(f) = \overline{I}(f)$.

**Theorem:** If $f$ is continuous in $\mathbb{R}^n$ on $Q$ then $f$ is integrable.

**Properties of the integral:**
Let $f, g : Q \subset \mathbb{R}^n \to \mathbb{R}$ be integrable and $\alpha, \beta \in \mathbb{R}$. Then:

1. $\alpha f + \beta g : Q \to \mathbb{R}$ is integrable and
$$\int_Q (\alpha f + \beta g)\, dx = \alpha \int_Q f(x)\, dx + \beta \int_Q g(x)\, dx$$
2. If $f(x) \leq g(x)$ $\forall x \in Q$, then
$$\int_Q f(x)\, dx \leq \int_Q g(x)\, dx$$
3. If $f(x) \geq 0$ $\forall x \in Q$, then $\int_Q f(x)\, dx \geq 0$
4. $$\left|\int_Q f(x)\, dx\right| \leq \int_Q |f(x)|\, dx \leq (\sup_Q |f(x)|) vol(Q)$$
5. Fubini's Theorem: Let $Q = I_1 \times \cdots I_n$, $f$ continuous on $Q$, then
$$\int_Q f(x_1, ..., x_n)\, dx_1 \cdots dx_n = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_n) dx_n \cdots dx_1$$
6. If $f = 1$, then $\int_Q 1\, dx = vol(Q)$

**Fubini's Theorem:** Suppose $D \subset \mathbb{R}^2$ a region which is of one of the following types:

1. $D_1 := \{(x, y) \,|\, a \leq x \leq b,\ g(x) < y < h(x)\}$
2. $D_2 := \{(x, y) \,|\, c \leq y \leq d,\ G(y) < x < H(y)\}$

- if $D$ is of type $D_1$, then
$$\int_D f(x, y)\, dxdy = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dxdy$$
- if $D$ is of type $D_2$, then
$$\int_D f(x, y)\, dxdy = \int_c^d \Big( \int_{G(y)}^{H(y)} f(x, y)\, dx \Big) dy$$

**Domain additivity:** If $X_1$ and $X_2$ are compact subsets of $\mathbb{R}^n$, and $f$ is continuous on $X_1 \cup X_2$, then

$$\int_{X_1 \cup X_2} f(x)\, dx + \int_{X_1 \cap X_2} f(x)\, dx = \int_{X_1} f(x)\, dx + \int_{X_2} f(x)\, dx.$$

In particular, if $X_1 \cap X_2$ is empty, then

$$\int_{X_1 \cup X_2} f(x)\, dx = \int_{X_1} f(x)\, dx + \int_{X_2} f(x)\, dx.$$

**Definition:** (1) Let $1 \le m \le n$ be an integer. A *parameterized m-set* in $\mathbb{R}^n$ is a continuous map $f : [a_1, b_1] \times \cdots \times [a_m, b_m] \to \mathbb{R}^n$ which is $C^1$ on $]a_1, b_1[ \times \cdots \times ]a_m, b_m[$.
(2) A subset $B \subset \mathbb{R}^n$ is *negligible* if there exists an integer $k \ge 0$ and parameterized $m_i$-sets $f_i : X_i \to \mathbb{R}^n$, with $1 \le i \le k$ and $m_i < n$, such that

$$X \subset f_1(X_1) \cup \cdots \cup f_k(X_k).$$

**Proposition:** Let $X \subset \mathbb{R}^n$ be a compact set. Assume that $X$ is negligible. Then for any continuous function on $X$, we have $\int_X f(x)\,dx = 0$.

## 3.3 Improper integrals

**Theorem:** Let $X \subset \mathbb{R}^n$ be a non-compact subset and $f : X \to \mathbb{R}$ a function s.t. $\int_K f(x)\,dx$ exists for every compact set $K \subset X$. Suppose we have s sequence of regions $X_k$, $k = 1, 2, ...,$ s.t.

1. Each region is closed and bounded
2. $X_{k+1} \supset X_k$
3. $\bigcup_{k=1}^{\infty} X_k = X$

Then, if $\lim_{n\to\infty} \int_{X_n} f(x)\,dx$ exists, we say that $\int_X f(x)\,dx$ *converges*, and

$$\int_X f(x)\,dx := \lim_{n\to\infty} \int_{X_n} f(x)\,dx.$$

## 3.4 The change of variable formula

We consider the analogue for the integral in $\mathbb{R}^n$ of the change of variable formula

$$\int f(g(x))g'(x)\,dx = \int f(y)\,dy$$

of one-variable calculus.

**Theorem:** Suppose we have $\phi : X \to Y$, where both $X$ and $Y$ are closed and bounded. We assume we can write

$$X = X_0 \cup B, \quad Y = Y_0 \cup C$$

where

1. the set $X_0$ and $Y_0$ are open
2. the sets $B$ and $C$ are negligible
3. $\phi : X_0 \to Y_0$ is a $C^1$ bijective map from $X$ to $Y$
4. $\det(J_p(x)) \neq 0\ \forall x \in X_0$

Let $Y = \phi(X)$ and $f : Y \to \mathbb{R}$ be continuous. Then

$$\int_Y f(y)\,dy = \int_X f(\phi(x))|\det J_p|\,dx.$$

### 3.4.1 Important examples

**1. Polar coordinates** (Integrating over a disc sector in $\mathbb{R}^2$)
$\phi : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$, $(r, \theta) \to (r\cos(\theta), r\sin(\theta))$;
$|\det(J_p(r, \theta))| = r$.

$$dxdy = rdrd\theta$$

**2. Cylindrical coordinates**
$\phi : [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $(r, \theta, z) \to (r\cos(\theta), r\sin(\theta), z)$;
$|\det(J_p)| = r$.

$$dxdydz = rdrd\theta dz$$

**3. Spherical coordinates** (Integrating over balls in $\mathbb{R}^3$)
$\phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$, $(r, \theta, \phi) \to (r\cos(\theta)\sin(\phi), r\sin(\theta)\cos(\phi), r\cos(\phi))$; $|\det(J_p)| = r^2\sin(\phi)$.

$$dxdydz = r^2 \sin(\phi) drd\theta d\phi$$

## 3.5 Geometric applications of integrals

**1. Center of mass:** Let $X \subset \mathbb{R}^n$ be bounded and clsoed. The center of mass of $X$ is the point where $X$ is perfectly balanced assuming uniform density. The coordinates of this point is $\tilde{x} = (\tilde{x_1}, ..., \tilde{x_n}) \in \mathbb{R}^n$ with

$$\tilde{x_i} = \frac{1}{vol(X)} \int_X x_i\,dx_1 dx_2 \cdots dx_n.$$

**2. Surface area in $n = 3$:** Suppose a surface in $\mathbb{R}^3$ is given by a function $f : \mathbb{R}^2 \to \mathbb{R}$, i.e. $S = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x, y) \in X,\ z = f(x, y)\}$. Then

$$\text{Area}(S) = \int\int_X \sqrt{1 + (\frac{\partial f}{\partial x}(x, y))^2 + (\frac{\partial f}{\partial y}(x, y))^2}\,dxdy.$$

## 3.6 The Green formula

**Definition:** A *simple closed parameterized curve* $\gamma : [a, b] \to \mathbb{R}^2$ is a closed parameterized curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or $\{s, t\} = \{a, b\}$, and such that $y'(t) \neq 0$ for $a < t < b$.

**Green's formula:** Let $X \subset \mathbb{R}^2$ be a compact set with a boundary $\partial X$ that is the union of finitely many simple closed parameterized curves $\gamma_1, ..., \gamma_k$. Assume that

$$\gamma_i : [a_i, b_i] \to \mathbb{R}^2$$

has the property that $X$ lies always "to the left" of the tangent vector $\gamma_i'(t)$ based at $\gamma_i(t)$. Let $f = (f_1, f_2)$ be a vector field of class $C^1$ defined on some open set containing $X$. Then we have

$$\int\int_X \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big) dxdy = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot d\vec{s}.$$

**Corollary:** Let $X \subset \mathbb{R}^2$ be a compact set with a boundary $\partial X$ that is the union of finitely many simple closed parametrized curves $\gamma_1, ..., \gamma_k$. Assume that

$$\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) : [a_i, b_i] \to \mathbb{R}^2$$

has the property that $X$ lies always "to the left" of the tangent vector $y_i'(t)$ based at $\gamma_i(t)$.
Then we have

$$\text{Vol}(X) = \sum_{i=1}^{k} \int_{\gamma_i} x \cdot d\vec{s} = \sum_{i=1}^{k} \int_{a_i}^{b_i} \gamma_{i,1}(t)\gamma_{i,2}'(d)\,dt.$$

**Example:** More generally, we can always use the Green formula to compute an integral over $X$. Indeed, for any function $g$, we can find many vector fields $f = (f_1, f_2)$ such that

$$g = \partial_x f_2 - \partial_y f_1.$$

For instance, we can put $f_1 = 0$ and find $f_2$ by solving $\partial_x f_2 = g$ (computing a primitive with respect to the $x$ variable).
As an example let $g(x, y) = (x^2 y^2)$ and let $X$ be the interior of an ellipse centered at 0 with axes lengths $a > 0$ in the $x$-direction and $b > 0$ in the $y$-direction. We want to compute

$$\int_X g(x, y)\,dxdy.$$

We put $f(x, y) = (0, \frac{1}{3}x^3y^2)$ to have $\partial_x f_2 = g$ and we parametrize the boundary by

$$\gamma(t) = (a\cos(t), b\sin(t)), \quad 0 \le t \le 2\pi,$$

which is a simple closed parametrized curve. So

$$\int_X g(x, y)\,dxdy = \int_\gamma f \cdot d\vec{s} = \frac{1}{3}a^3b^2 \int_0^{2\pi} \cos^3(t)\sin^2(t) \times b\cos(t)dt.$$

Using trigonometric computations, we find that

$$\int_0^{2\pi} \cos^4 \sin^2\ dt = \frac{\pi}{8},$$

so the integral is $\pi a^3 b^3/24$.

# 4 Random

## 4.1 Different methods

(1) Compute in which direction a directional derivative is largest at a given point:

1. Compute gradient (partial derivatives with respect to each variable)
2. Plug variables of given point into gradient vector to get response vector

(2) Compute Eigenvalues of a matrix

1. Compute the polynomial $\det(A - \lambda I)$
2. The roots $\lambda_i$ of the above polynomial are the eigenvalues of $A$

(3) Compute determinante of $2 \times 2$ matrix $A$

$$\det(A) = a \cdot d - c \cdot b$$

(4) Compute determinante of $3 \times 3$ matrix $A$

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh$$

(5) Jacobi decomposition

$$d(h \circ f)(a) = dh(f(a)) \cdot df(a)$$

(6) Compute line integral $\int_\gamma f ds$

1. Check if $f$ is conservative, if yes, find potential $g$ and use $\int_\gamma f ds = g(\gamma(b)) - g(\gamma(a))$
2. Use $\int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$
3. Use other properties of the line integral (e.g. closed loop = 0)
4. Use Green's formula

(7) Usage of Green's Theorem

1. Compute the area of a region as a line integral by taking the vector field to be $f(x, y) = (0, x)$
2. Calculate a line integral if the double integral of the *curl* of $f$ looks easier, especially if $curl(f) = 0$
3. $\int\int_X 1 dxdy = Area(X) \to$ if we can find $f : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $curl(f) \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 1$, we can take $f = (0, x)$ or $f = (-y, 0)$.

(8) Calculating the directional derivative of $f$ along $v$ at a point $x_0 \in \mathbb{R}^n$

1. Use the definition, i.e. the differential quotient

$$D_v f(x_0) = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

2. Define $\phi : [-\delta, \delta] \to \mathbb{R}$, $t \to \phi(t) = f(x_0 + tv)$ and use

$$\phi'(0) = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

3. For $m = 1$ we can use

$$D_v f(x_0) = \vec{v} \cdot df(x_0) = v_1 \cdot \partial_1 f(x_0) + v_2 \cdot \partial_2 f(x_0) + \cdots$$

which means that the directional derivative is a linear combination of the partial derivatives of $f$.

## 4.2 Identities

### 4.2.1 Angles

| **deg** | $0°$ | $30°$ | $45°$ | $60°$ | $90°$ | $180°$ |
|---|---|---|---|---|---|---|
| **rad** | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ |
| **cos** | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ |
| **sin** | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ |
| **tan** | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | $+\infty$ | $0$ |

### 4.2.2 Trigonometric Identities

- $e^{ix} = \cos(x) + i\sin(x)$
- $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
- $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
- $\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}$
- $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

## 4.3 Differentials and Derivatives

| **f'(x)** | **f(x)** | **F(x)** |
|---|---|---|
| $\cosh(x)$ | $\sinh(x)$ | $\cosh(x) + C$ |
| $\cos(x)$ | $\sin(x)$ | $-\cos(x)$ |
| $2\sin(x)\cos(x)$ | $\sin^2(x)$ | $\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right)$ |
| $-\sin(x)$ | $\cos(x)$ | $\sin(x)$ |
| $-2\sin(x)\cos(x)$ | $\cos^2(x)$ | $\frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right)$ |
| $\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$ | $\tan(x)$ | $-\ln\lvert\cos(x)\rvert$ |
| $\frac{1}{\cosh^2(x)}$ | $\tanh(x)$ | $\log(\cosh(x))$ |
| $-\frac{1}{\sin^2(x)}$ | $\cot(x)$ | $\ln\lvert\sin(x)\rvert$ |

# 5 Exercises

## 5.1 Differential Equations

**a)** $y''' - 4y'' + 6y' = e^{2t} + 9t^2$
We have constant coefficients so the solution must be of the form $e^{\lambda t}$. We calculate the roots of the char. poly.:

$$P(\lambda) = \lambda^3 - 4\lambda^2 + 6\lambda = \lambda(\lambda^2 - 4\lambda + 6)$$

$\Rightarrow \lambda_1 = 0$, $\lambda_{2,3} = \frac{4 \pm \sqrt{-8}}{2} = 2 \pm i\sqrt{2}$.
The general solution is therefore given by:

$$y_h(t) = c_1 e^{0t} + c_2 e^{(2+i\sqrt{2})t} + c_3 e^{(2-i\sqrt{2})t}$$
$$= c_1 + e^{2t}(c_2 \sin(\sqrt{2}t) + c_3 \cos(\sqrt{2}t))$$

To find the inhomo. sol. we split the diff. eqn. into two parts (*principle of superposition*), i.e.

1. $y''' - 4y'' + 6y' = e^{2t}$
2. $y''' - 4y'' + 6y' = 9t^2$

For (1) we take as Ansatz $y_{p1}(t) = ae^{2t}$ and calculate the derivatives:

- $y'_{p1}(t) = 2ae^{2t}$
- $y''_{p1}(t) = 4ae^{2t}$
- $y'''_{p1}(t) = 8ae^{2t}$

We substitute into the ODE:

$$e^{2t}(8a - 16a + 12a) = e^{2t} \Rightarrow e^{2t}(4a) = e^{2t} \Rightarrow a = \frac{1}{4}$$

We get for our first solution $y_{p1}(t) = \frac{1}{4}e^{2t}$.
For (2) we take as Ansatz $y_{p2}(t) = at^3 + at^2 + ct$. Note that $a)$ since there is no $y$ in our ODE, we need a 3rd degree poly. to get a $t^2$ from $y'$, and $b)$ since 0 is a root of our char. poly., we cannot have a constant $+d$ in our Ansatz since that wouldn't allow for 0 to be a root. We again calculate the derivatives:

- $y'_{p2}(t) = 3at^2 + 2bt + c$
- $y''_{p2}(t) = 6at + 2b$
- $y'''_{p2}(t) = 6a$

We again substitute into the ODE and get:

$$\underbrace{18a}_{=9} t^2 + \underbrace{(12b - 24a)}_{=0} t + \underbrace{(6a - 8b + 6c)}_{=0} = 9t^2$$

By backward substitution we arrive at $a = \frac{1}{2}$, $b = 1$ and $c = \frac{5}{6}$ from where we get our second solution $y_{p2}(t) = \frac{1}{2}t^3 + t^2 + \frac{5}{6}t$. By the superposition principle, $y_p(t) = y_{p1} + y_{p2}$ and we have the general solution of our ODE by $y(t) = y_h(t) + y_p(t)$:

$$c_1 + e^{2t}(c_2 \sin(\sqrt{2}t) + c_3 \cos(\sqrt{2}t)) + \frac{1}{4}e^{2t} + \frac{1}{2}t^3 + t^2 + \frac{5}{6}t, \ c_1, c_2, c_3 \in \mathbb{R}$$

---

**b)** $y = 2t^2 y'$, $t \geq 1$, with $y'(1) = 1$
We notice that the equation is *separable:*

$$y = 2t^2 y \Rightarrow \frac{y'}{y} = \frac{1}{2t^2} \Rightarrow \int \frac{1}{y} dy = \int \frac{1}{2t^2} dt$$

By integrating both sides we get:

$$\log(|y|) = -\frac{1}{2t} + c \Rightarrow |y| = \exp(-\frac{1}{2t} + c) \Rightarrow y = Ke^{-\frac{1}{2t}}$$

Now, with our initial condition $y'(1) = 1$, we get:

$$y'(1) = Ke^{-\frac{1}{2}} \stackrel{!}{=} 1 \Rightarrow K = e^{\frac{1}{2}}$$

Finally, the general solution to our diff. eqn. is given by $y(t) = \exp(1/2 - 1/2t)$.

**c)** $y' = y^2 - 1$
We first notice that the RHS only vanishes if $y(x) = \pm 1$. In fact we see, that the constant functions $y_{1,2}(x) = \pm 1$ are both solutions ot the problem.
If $y(x) \neq \pm 1$, we can use separation of variables:

$$y' = y^2 - 1 \Rightarrow \frac{y'}{y^2-1} = 1 \Rightarrow \int \frac{1}{y^2-1} dy = \int 1 dx = x + c.$$

To calculate $\int \frac{1}{y^2-1} dy$ we use partial decomposition:

$$\frac{1}{y^2-1} = \frac{A}{y+1} + \frac{B}{y-1}.$$

By multiplying $A$ by the denom. of the 2nd fraction and $B$ by the denom. of the 1st, we get

$$1 = (y-1)A + (y+1)B \Rightarrow 1 == 1(B-A) + \overset{=0}{yA + yB}$$

It follows that $B = \frac{1}{2}$ and $A = -\frac{1}{2}$ and therefore $\frac{1}{y^2-1} = \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$. With this decomposition we can calculate the integral from above:

$$\int \frac{1}{y^2-1} = \frac{1}{2}\int \frac{1}{y-1} dy - \frac{1}{2}\int \frac{1}{y+1} dy = \frac{1}{2}(\log(y-1) - \log(y+1))$$

From here it follows that

$$\frac{1}{2}\log\Big(\frac{y-1}{y+1}\Big) = x + c \Rightarrow \frac{y-1}{y+1} = e^{2c}e^{2x}$$

After elementary transformations we get the solution

$$y = \frac{Ce^{2x}+1}{1-Ce^{2x}}, \quad C \in \mathbb{R}.$$

## 5.2 Multivariable Calculus

**a)** Determine the critical points of the function
$f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 + y^3 + 3xy$
We first calculate $Df$:

$$Df = (3x^2 + 3y, 3y^2 + 3x)$$

Then we search for the points s.t. $Df = 0$, i.e. $3x^2 + 3y = 0$ and $3y^2 + 3x = 0$. We therefore have $x^2 = -y$ and $y^2 = -x$ which results in the two equations $y(y^3 + 1) = 0$ and $x(x^3 + 1)$.
The polynomial $X(X^3 + 1) = 0$ has the real solutions 0 and $-1$ and therefore we have to consider the two critical points $(-1, -1)$ and $(0, 0)$. We then calculate $\text{Hess}_f$ to determine the type of the two critical points:

$$\text{Hess}_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix} = \begin{pmatrix} 6x & 3 \\ 3 & 6y \end{pmatrix}$$

For $(0, 0)$, we have $\text{Hess}_f(0, 0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$. We calculate the eigenvalues:

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 3 \\ 3 & -\lambda \end{pmatrix} = B \to \det(B) = (\lambda^2 - 9)$$

We have the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -3$, and therefore $(0, 0)$ is a saddle point.

For $(-1, -1)$ we have $\text{Hess}_f(-1, -1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$. The eigenvalues, calculated the same way as above, are $\lambda_1 = -1$ and $\lambda_2 = -2$. Therefore, $(-1, -1)$ is a local maximum.

**b)** Determine the value of the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy dx$.
Since $0 \le x \le 1$ and $x^2 \le y \le 1$, it follows that $0 \le y \le 1$ and $0 \le x \le \sqrt{y}$. We therefore have

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) dy dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3) dx dy$$

It follows that

$$\begin{aligned}\int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3) dx dy &= \int_0^1 \Big[\frac{x^4}{4}\sin(y^3)\Big]_0^{\sqrt{y}} dy \\ &= \int_0^1 \Big(\frac{1}{4}(\sqrt{y})^4 \sin(y^3) - \frac{1}{4}\cdot 0 \cdot \sin(y^3)\Big) dy \\ &= \frac{1}{4}\int_0^1 y^2 \sin(y^3) dy = \frac{1}{4}\Big[-\frac{1}{3}\cos(y^3)\Big]_0^1\end{aligned}$$

Which yields the solution $-\frac{1}{12}(\cos(1) - 1)$.

**c)** The directional derivative of the function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = ze^{\cos(xy)}$ at the point $a = (1, \pi/2, 1)$ is largest in which direction?
The directional derivative is largest in the direction of the gradient itself. We therefore need to calculate the gradient first:

$$\nabla f = \begin{pmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{pmatrix} = \begin{pmatrix} -y\sin(xy) z e^{\cos(xy)} \\ -x \sin(xy) z e^{\cos(xy)} \\ e^{\cos(xy)} \end{pmatrix}$$

We then calculate $\nabla f(a)$ to get the direction in which the directional derivative is largest:

$$\nabla f(a) = \nabla f \begin{pmatrix} 1 \\ \frac{\pi}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2}\sin(\pi/2) 1 e^{\cos(\pi/2)} \\ -1 \sin(\pi/2) 1 e^{\cos(\pi/2)} \\ e^{\cos(\pi/2)} \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2} \\ -1 \\ 1 \end{pmatrix}$$

**d)** Given the vectorfield $v : \mathbb{R}^2 \to \mathbb{R}^2$,

$$v(x, y) = \begin{pmatrix} x^2 + y^2 \\ 2xy + 2y \end{pmatrix}.$$

1. Show that $v$ is conservative.
Since $\mathbb{R}^2$ is star-shaped and open and $v \in C^1$, it is enough to show that $\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}$:

$$\frac{\partial v_1}{\partial y} = \frac{1}{\partial y}(x^2 + y^2) = 2y = \frac{1}{\partial x}(2xy + 2y) = \frac{\partial v_2}{\partial x}.$$

2. Compute the path integral $\int_\gamma v ds$, where the piece-wise continuously differentiable curve $\gamma : [0, 6] \to \mathbb{R}^2$ is given by

$$\gamma(t) = \begin{cases} (t-1, 1), & \text{if } t \in [0, 2] \\ (3-t, 3-t), & \text{if } t \in [2, 4] \\ (-1, t-5), & \text{if } t \in [4, 6] \end{cases}.$$

Since $v(x, y)$ is conservative, we know that the line integral of any closed curve is 0. Indeed, $\gamma$ is a closed curve, since $\gamma_1(0) = (-1, 1) = \gamma_3(6)$.

**e)** Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$v(x, y) = \begin{pmatrix} 3 \\ 3y^3 x + yx^3 \end{pmatrix}.$$

Compute the path integral $\int_\gamma v\, ds$, where $\gamma$ is the boundary of the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le y \text{ and } x^2 + y^2 \le 4\},$$

parametrized in the counterclockwise direction.
We know that $\int_\gamma v ds = \int_\Omega \Big(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\Big) dx dy$. In our case:

$$\int_\Omega \Big(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\Big) dx dy = \int_\Omega 3y^3 + 3yx^2 dx dy = 3\int_\Omega y^3 + yx^2 dx dy.$$

If we use polar coordinates, i.e. $x = r\cos(\phi)$ and $y = r\sin(\phi)$, with $0 \le r \le r$ and $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$, we get:

$$3\int_0^2 \int_{\pi/4}^{\pi/2} r(r^3 \sin^3(\phi) + r^3 \sin(\phi)\cos^2(\phi)) d\phi dr$$

$$= 3\int_0^2 r^4 dr \int_{\pi/4}^{\pi/2} \sin(\phi)(\sin^2(\phi) + \cos^2(\phi))d\phi$$

$$= 3\int_0^2 r^4 dr \int_{\pi/4}^{\pi/2} \sin(\phi)d\phi$$

Evaluating this integral gives us:

$$3\Big[\frac{r^5}{5}\Big]_0^2\Big[-\cos(\phi)\Big]_{\pi/4}^{\pi/2} = 3\cdot\frac{32}{5}\cdot\cos\Big(\frac{\pi}{4}\Big) = \frac{48\sqrt{2}}{5}$$

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**f)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = e^x \sin(y)$. Determine the first and second degree Taylorpolynomials of $f$ at the point $(x_0, y_0) = (0, \frac{\pi}{2})$. Approximate the value of $f$ at the point $(0, \frac{\pi}{2} + \frac{1}{4})$ using the second order Tayplorpolynomial.

First we need to compute all first and second order partial derivatives:

$$\frac{\partial f}{\partial x} = e^x \sin(y),\ \frac{\partial f}{\partial y} = e^x \cos(y)$$

$$\frac{\partial^2 f}{\partial^2 x} = e^x \sin(y),\ \frac{\partial^2 f}{\partial^2 y} = -e^x \sin(y),\ \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = e^x\cos(y)$$

Now we let $y := \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \begin{pmatrix} x \\ y - (\pi/2) \end{pmatrix}$ and $x_0 := \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. The compact way of writing the first and second degree Taylorpolynomials is:

$$T_1 f(y; x_0) = f(x_0) + y^T \nabla f(x_0),$$

$$T_2 f(y; x_0) = f(x_0) + y^T \nabla f(x_0) + y^T H_f(x_0) y$$

From this we get the two Taylorpolynomials as:

$$T_1 f = 1 + (x,\ y - \frac{\pi}{2}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + x$$

$$T_2 f = 1 + x + \frac{1}{2}(x,\ y - \frac{\pi}{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y - \frac{\pi}{2} \end{pmatrix}$$

$$= 1 + x + \frac{1}{2}\Big(x^2 - \Big(y - \frac{\pi}{2}\Big)^2\Big)$$

Plugging $(0,\ \frac{\pi}{2} + \frac{1}{4})$ into $T_2 f$ we get

$$f\Big(0,\ \frac{\pi}{2} + \frac{1}{4}\Big) \simeq 1 - \frac{1}{2}\Big(\frac{\pi}{2} + \frac{1}{4} - \frac{\pi}{2}\Big)^2 = 1 - \frac{1}{32} = \frac{31}{32}.$$

## 5.3 Multiple Choice

(1) If $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is such that $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$, all exist and are continuous, then $f$ is differentiable on $\mathbb{R}^3$. **False:** For the statement to be true, for each variable $\partial x_i$, each partial derivative $\partial f_i$ needs to exist and needs to be continuous.
(2) If $f : \mathbb{R} \to \mathbb{R}^2$ is of class $C^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is of class $C^1$, then $g \circ f$ is of class $C^1$. **True.**
(3) Let $D$ be the disc of radius 1 centered at 0 in $\mathbb{R}^2$ and let $A$ be an invertible matrix of size 2 with real coefficients. The area of $A(D)$ is equal to $\det(A)$. **False:** The area of $D$ is equal to $\pi \cdot\cdot\cdot |\det(A)|$.
(4) If $f$ is a vector field of class $C^1$ on $\mathbb{R}^2 - \{0\}$ and $\int_\gamma f \cdot d\vec{s} = 0$ for all closed curves $\gamma : [0, 1] \to \mathbb{R}^2 - \{0\}$, then $f$ is conservative.
(5) The function on $\mathbb{R}^3$ defined by

$$f(x, y, z) = xyz - x^2 y + \pi z^2 + y^2 z^2$$

is a polynomial of degree 3. **False.**
(6) Which of the following vector fields on $\mathbb{R}^2$ are conservative?

1. $(3x^2 + y,\ x + 4y^3)$
2. $(\sin(x),\ \cos(y))$
3. $(\sin(y),\ \cos(x))$
4. $\Big(\frac{x}{1+x^2+y^4},\ \frac{4y^3}{1+x^2+y^4}\Big)$

**1 and 2 are conservative:** We need to check for each vector field, whether or not $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$. If this is the case, the field is conservative.
(7) If $f_1$ and $f_2$ are solutions to the differential equation $y'' - xy' + y = \cos(x)$, then so is $f_1 + 2f_2$. **False:** Would only hold if the differential equation would be homogeneous.
(8) If $f$ is of $C^2$ on $\mathbb{R}^2$ and $f$ is maximal at $(x_0, y_0)$, then $\frac{\partial^2 f}{\partial^2 x}(x_0, y_0) = 0$. **False.**
(9) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^n$, $n \geq 1$, $f(x) = (f_1(x), ..., f_n(x))$ be a function. Then

- If one of the $f_j$'s is injective, then $f$ is injective. **True:** Can be shown by $x_1 \neq x_2 \Rightarrow f_1(x_1) \Rightarrow f_1(x_2) \Rightarrow f(x_1) \neq f(x_2)$
- If $f$ is injective, at least one of the $f_j$'s is injective. **False:** Consider $f(x) = (\cos(x), \sin(x))$ in $[0, 2\pi)$.
- If every $f_j$ is surjective, then $f$ is surjective. **False:** Consider $f(x) = (x, x, ..., x)$ is surjective for each $f_j$, but $(1, 0, 0, ..., 0) \notin f(\mathbb{R})$.
- If $f$ is surjective, then every $f_j$ is surjective. **True**

(10) Let $f : [a, b] \to \mathbb{R}^n$, $n \geq 1$, be continuous and differentiable in $(a, b)$. Then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**False:** Consider $f(x) = (\cos(x), \sin(x))$ in $[0, 2\pi)$.
(11) Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear function. Then if $L$ is continuous at one point $x_0 \in \mathbb{R}^m$, it is continuous at every point in $\mathbb{R}^m$. **True**
(12) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0, 0) = 0$. For $f$ to be continuous at $(0, 0)$, the fact that $\lim_{x\to 0} f(x, 0) = \lim_{y\to 0} = 0$ is necessary, but in general not sufficient. **True:** Consider the following counterexample:

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

(13) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. For $f$ to be continuous at $(0, 0)$

- it is sufficient that, along some direction $v \neq 0$, the directional derivative $D_v f(0, 0)$ exists. **False**
- it is sufficient that both the partial derivatives $\partial_x f(0, 0)$ and $\partial_y f(0, 0)$ exist. **False**
- it is sufficient that the directional derivatives $D_v f(0, 0)$ along every direction $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ exist. **False**

The same counterexample as in (12) applies here.
(14) Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, regular region. Generally speaking, the integral $\int_\Omega d\mu$ represents

- the area of $\Omega$. **True**
- the length of the curve bounding $\Omega$. **False**
- the volume of some cylinder with base $\Omega$ and height 1. **True**
- the surface area of some cylinder with base $\Omega$ and height 1. **False**

(15) Let $\Omega \subset \mathbb{R}^n$ be an open set and consider its measure $|\Omega| = \int_\Omega dx_1 ... dx_n$. Then:

- If $\Omega$ is unbounded, then the integral is divergent. **False:** Take $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, x > 1,\ 0 < y < \frac{1}{x^2}\} \subset \mathbb{R}^2$. Then:

$$|\Omega| = \int_1^\infty \frac{dx}{x^2} = 1,$$

  so convergent.
- If the integral is divergent, then $\Omega$ is unbounded. **True**
- If there exists $\epsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ s.t. $B_\epsilon(x_j) \subseteq \Omega$ for every $j \in \mathbb{N}$, then the integral is divergent. **True**
- If the integral is unbounded, then there exists $\epsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ s.t. $B_\epsilon(x_j) \subseteq \Omega$ for every $j \in \mathbb{N}$. **False**

(16) Let $f : \mathbb{R}^n \to [0, \infty)$ be a non-negative, continuous function. Then:

- If $\lim_{x\to\infty} f(x) = 0$, the improper integral $\int_{\mathbb{R}^n} f dx$ exists and is finite. **False:** Consider $f(x) = \frac{1}{1+|x|}$.
- If the improper integral $\int_{\mathbb{R}^n} f dx$ exists and is finite, then $\lim_{x\to\infty} f(x) = 0$. **False**
- If $\lim_{x\to\infty} f(x)$ exists and is nonzero, the improper integral $\int_{\mathbb{R}^n} f dx$ is non-finite.